

**GENERALIZED ZETA FUNCTION REPRESENTATION OF
GROUPS AND 2-DIMENSIONAL TOPOLOGICAL
YANG-MILLS THEORY:
THE EXAMPLE OF $GL(2, \mathbb{F}_q)$ AND $PGL(2, \mathbb{F}_q)$.**

PH. ROCHE

ABSTRACT. We recall the relation between Zeta function representation of groups and two-dimensional topological Yang-Mills theory through Mednikh formula. We prove various generalisations of Mednikh formulas and define generalization of Zeta functions representations of groups. We compute some of these functions in the case of the finite group $GL(2, \mathbb{F}_q)$ and $PGL(2, \mathbb{F}_q)$. We recall the table characters of these groups for any q , compute the Frobenius-Schur indicator of their irreducible representations and give the explicit structure of their fusion rings.

1. INTRODUCTION

2-dimensional Yang-Mills theory associated to compact real Lie groups is now a well studied subject initially developped in [22][30]. In the case where the gauge group G is a connected simply connected compact Lie group (for example $SU(n)$), the partition function of 2-dimensional Yang Mills theory on a Riemann surface Σ of genus g can be computed explicitly and is equal to

$$(1) \quad \mathcal{Z}_G(\lambda, g) = \sum_{\pi \in \hat{G}} \frac{1}{(\dim \pi)^{2g-2}} \exp(-\lambda a(\Sigma) C(\pi))$$

where the summation is over the set of classes of irreducible finite dimensional complex representations of G , λ denotes the Yang-Mills coupling constant, $a(\Sigma)$ the area of Σ and $C(\pi)$ the value of the Casimir element of π .

In the limit where λ goes to zero, the so-called topological limit, the partition function depends only on the genus and is equal to:

$$(2) \quad \mathcal{Z}_G(g) = \sum_{\pi \in \hat{G}} \frac{1}{(\dim \pi)^{2g-2}}.$$

The initial motivation of the present work was to study low dimensional field theory in the case where the group is a p -adic one and to study generalisation of the previous formulas in the context of p -adic compact Lie group (such as for example $SL(n, \mathbb{Z}_p)$). This could be the initial step to build

adelic low dimensional field theory, theories which could be of fundamental importance. See Y.Manin [21] for reflections on this topic and [6] for a review of p -adic and adelic physics.

The formula (2) gives a bridge with the study of the behaviour of dimensions of representations of groups. Let G be a compact topological group, if π is a continuous complex representation of necessarily finite dimension $\dim \pi$, we denote χ_π its character and V_π the vector space on which π acts.

Let \widehat{G} be the set of isomorphism classes of irreducible finite dimensional complex representations. We assume that for each positive integer n the number of elements of \widehat{G} having dimension n is finite of cardinal a_n . Groups having these properties are called rigid [28].

One can define the ζ_G function representation of G by

$$(3) \quad \zeta_G(s) = \sum_{n \geq 1} \frac{a_n}{n^s} = \sum_{\pi \in \widehat{G}} \frac{1}{(\dim \pi)^s}, \quad s \in \mathbb{C}.$$

In the case where G is a connected simply connected real compact Lie group, G is rigid and the function ζ_G is completely determined by the root system of G . Fix a Cartan torus T of G , it defines a root system Φ , fix a choice Φ^+ of polarization of Φ , there is a one to one correspondence between \widehat{G} and the set of positive integer weights Λ^+ . If $\lambda \in \Lambda^+$, it defines an irreducible representation π_λ of G which dimension is given by the formula $\dim(\pi_\lambda) = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle}$ where ρ is half of the sum of positive roots. The ζ_G representation function has therefore the simple expression:

$$(4) \quad \zeta_G(s) = \sum_{\lambda \in \Lambda^+} \prod_{\alpha \in \Phi^+} \left(\frac{\langle \alpha, \rho \rangle}{\langle \alpha, \lambda + \rho \rangle} \right)^s, \quad s \in \mathbb{C}.$$

It has been shown in [20] that the abscisse of convergence of ζ_G , i.e the infimum of $\alpha > 0$ for which $\zeta_G(s)$ is absolutely convergent for $\operatorname{Re}(s) > \alpha$, is given by $2/h$ where h is the Coxeter number of G .

In the case of p -adic compact Lie group the situation is much more complicated. This comes from the fact that the classification of finite dimensional irreducible complex representations of G is more involved and uses an analog of Kirillov Theory. For a review of known results see [16, 17, 28]. When G is a compact p -adic Lie group which is FAb, i.e $H/[H, H]$ is finite for every open subgroup H of G , then G is rigid. A fundamental theorem, due to A.Jaikin-Zapirain [14], asserts that, under the assumption that G is a compact p -adic Lie group which is FAb, there exists integers n_1, \dots, n_k and rational functions f_1, \dots, f_k such that

$$(5) \quad \zeta_G(s) = \sum_{j=1}^k \frac{f_j(p^{-s})}{n_j^s}.$$

In some very few cases one can even compute exactly ζ_G , this is the case for $SL(2, \mathcal{O})$ [14] where \mathcal{O} is the ring of integers of a p -adic field and more recently for $SL(3, \mathcal{O})$ [3].

In the first part of this article we will review results of interest on ζ_G function representations and introduce other functions generalizing these ζ_G functions which are connected to correlation functions of 2D topological Yang Mills theory field theory and its generalisations. These functions are still actively studied in the real case but have not received any attention in the p -adic context.

In the second part of this article we compute explicitly these functions for the simplest case where G is a finite group of type $GL(2, \mathbb{F}_q)$ and $PGL(2, \mathbb{F}_q)$. Although these functions are very simple compared to those associated to p -adic compact Lie groups, they are already non trivial and are the preliminary step for computing the generalized functions for p -adic Lie groups of the type $GL(2, \mathcal{O}), PGL(2, \mathcal{O})$, where \mathcal{O} is the ring of integer of a p -adic field. This is the subject of a subsequent work [4].

We also collect in an appendix the character table of the groups $GL(2, \mathbb{F}_q)$, $PGL(2, \mathbb{F}_q)$ enriched by the list of Frobenius-Schur indicators as well as the explicit decomposition of tensor product of irreducible representations. This last result, although being straightforwardly computed from the character table of the groups, has not appeared in such a detailed form in the litterature (see however [2] [15]).

2. GENERALIZED ZETA FUNCTION REPRESENTATIONS OF GROUPS

2.1. Zeta function representation. In the case where G is finite, the value of ζ_G on positive even integer has a nice geometrical interpretation. We will use this interpretation to define new zeta functions which will have the property that evaluated on some integers they will also have nice geometrical interpretation.

Let G be a finite group, let Σ be a compact connected orientable topological surface of genus $g \geq 0$. We fix a point x on Σ , and consider the set of group morphisms $X_G(x, \Sigma) = Hom(\pi_1(x, \Sigma), G)$.

Because the groups $\pi_1(x, \Sigma)$ do not depend on x up to isomorphism, we will not mention x in a context where results do not depend on the choice of x .

G acts on $X_G(x, \Sigma)$ by conjugation and we denote $X_G(x, \Sigma)/AdG$ the set of equivalence classes. $X_G(x, \Sigma)/AdG$ is in bijection with the set of G -principal bundle over Σ . This set is also called the character variety of Σ .

The formula of Mednykh [23] gives:

$$(6) \quad \frac{|X_G(x, \Sigma)|}{|G|} = |G|^{-\chi(\Sigma)} \sum_{\pi \in \hat{G}} (dim \pi)^{\chi(\Sigma)},$$

where $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Σ .

We recall here a simple proof of this formula.

Proof:

Σ is homeomorphic to the connected sum $\Sigma \simeq T^{\#g}$ where T is the two dimensional torus. Therefore $\pi_1(x, \Sigma)$ has a presentation by generators $a_1, \dots, a_g, b_1, \dots, b_g$ and relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = e$.

We therefore obtain

$$X_G(x, \Sigma) = \{(A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g}, \prod_{i=1}^g [A_i, B_i] = e\}.$$

We denote θ_T the function $\theta : G \rightarrow \mathbb{N}, \theta_T(k) = |\{(A, B) \in G^2, ABA^{-1}B^{-1} = k, \}|$. As a result

$$\begin{aligned} |X_G(x, \Sigma)| &= |G|^{2g} \int_{G^{2g}} \delta\left(\prod_{i=1}^g [A_i, B_i]\right) \prod_i dA_i \prod_i dB_i \\ &= |G|^g \int_{G^g} \theta_T(k_1) \cdots \theta_T(k_g) \delta(k_1 \cdots k_g) dk_1 \cdots dk_g \\ &= |G|^{g-1} \underbrace{(\theta_T * \cdots * \theta_T)}_{g \text{ times}}(e) \end{aligned}$$

where \int is the normalized Haar measure on G , δ is the delta function at e defined by $\delta(x) = 1$ if $x = e$ and zero otherwise, and $*$ denotes the convolution product, $(\phi * \psi)(x) = \int_G \phi(xy) \psi(y^{-1}) dy$.

The function θ_T admits the following Fourier decomposition:

$$(7) \quad \theta_T = |G| \sum_{\pi \in \hat{G}} \dim(\pi)^{-1} \chi_\pi,$$

which follows from $|G|\delta = \sum_{\pi \in \hat{G}} \dim(\pi) \chi_\pi$ and from the formula

$$(8) \quad \int_G \langle \phi, \pi(g)(v) \rangle \langle \psi, \pi(g^{-1})(w) \rangle dg = \frac{\langle \phi, w \rangle \langle \psi, v \rangle}{\dim \pi}, \forall v, w \in V_\pi, \forall \phi, \psi \in V_\pi^*.$$

Using the Fourier decomposition of θ_T and the identity $\chi_\pi * \chi_{\pi'} = \frac{1}{\dim \pi} \delta_{\pi, \pi'} \chi_\pi$, we obtain Mednykh formula. \square

Note also that $\zeta_G(0) = |\hat{G}|$ which is also equal to the number of conjugacy classes of G . We also have $\zeta_G(-2) = |G|^2$.

When G is a connected simply connected semisimple compact Lie group, an analogue of this correspondence between the geometry of G -principal bundles and zeta function representation exists and was first obtained by E. Witten [30]: the evaluation of ζ_G on the integers $2g-2$ for $g \geq 2$ gives the volume of the moduli space of G -flat connections on a compact orientable topological surface of genus g . A rigorous mathematical proof of it was given for example in [24], one has:

$$(9) \quad \text{Vol}(X_G(x, \Sigma)/\text{Ad}G) = |Z(G)| (\text{Vol}(G))^{2g-2} \zeta_G(2g-2)$$

where $Z(G)$ is the center of G , $\text{Vol}(G)$ is the Riemannian volume of G computed using the invariant metric obtained from the Killing form.

Note that it is $X_G(x, \Sigma)/\text{Ad}G$ which appear and not $X_G(x, \Sigma)$.

These results can be generalized to the case of surfaces with boundaries, which is the content of the next section.

2.2. Generalized zeta functions and surfaces with boundaries. Let G be a finite group, the ζ_G function can be generalized to the case where one consider connected compact orientable topological surfaces with boundaries. Let Σ be a connected compact orientable topological surface Σ of genus g and define $\Sigma' = \Sigma \setminus (\cup_{j=1}^r D_j)$ where D_j are open disks included in Σ with $(\overline{D_j})$ pairwise non intersecting. We have $\partial\Sigma' = C_1 \cup \dots \cup C_r$, where C_1, \dots, C_r are homeomorphic to circles. Let x be an interior point of Σ' , and fix closed curves ℓ_1, \dots, ℓ_r originating from x such that the homotopy class of ℓ_j is the same as C_j . The homotopy group $\pi(\Sigma', x)$ is isomorphic to the group generated by $a_i, b_i, m_j, i = 1, \dots, g, j = 1, \dots, r$, with relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^r m_j = e$.

Let denote $\gamma_1, \dots, \gamma_r$ elements of G , the set of principal G -bundles over Σ' such that the holonomy along ℓ_j belongs to the conjugacy class of $\gamma_j, j = 1, \dots, r$, is a finite set denoted $X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)$. If $\gamma \in G$ we denote $O(\gamma)$ the conjugacy class of γ . The characteristic function of $O(\gamma)$ is denoted $\delta_{O(\gamma)}$ and admits the expansion:

$$(10) \quad \frac{|G|}{|O(\gamma)|} \delta_{O(\gamma)} = \sum_{\pi \in \hat{G}} \chi_\pi(\gamma^{-1}) \chi_\pi.$$

Mednykh formula can be generalized as:

$$(11) \quad \frac{|X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)|}{|G|} = |G|^{2g-2} \prod_{j=1}^r |O(\gamma_j)| \sum_{\pi \in \hat{G}} \frac{\prod_{j=1}^r \chi_\pi(\gamma_j)}{(\dim \pi)^{2g-2+r}}.$$

Proof:

It is a consequence of

$$X_G(x, \Sigma, \gamma_1, \dots, \gamma_r) =$$

$$\{(A_1, \dots, A_g, B_1, \dots, B_g, M_1, \dots, M_r) \in G^{2g+r}, \prod_{i=1}^g [A_i, B_i] \prod_{j=1}^r M_j = e, \forall j = 1, \dots, r, M_j \in O(\gamma_j)\},$$

hence

$$\begin{aligned} & |X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)| = \\ & |G|^{2g+r} \int_{G^{2g+r}} \delta\left(\prod_{i=1}^g [A_i, B_i] \prod_{j=1}^r M_j\right) \prod_{j=1}^r \delta_{O(\gamma_j)}(M_j) \prod_i dA_i \prod_i dB_i \prod_j dM_j. \end{aligned}$$

The formula (11) is obtained by using the expansion (10) of the characteristic function of conjugacy class and using repetitively the formula (8). \square

Therefore we are led to define, for a rigid compact group G , a generalized zeta functions $\zeta_G^{(r)} : G^r \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$(12) \quad \zeta_G^{(r)}(\gamma_1, \dots, \gamma_r)(s) = \sum_{\pi \in \widehat{G}} \frac{\prod_{j=1}^r \chi_\pi(\gamma_j)}{(\dim \pi)^{s+r}}, s \in \mathbb{C}, \gamma_1, \dots, \gamma_r \in G.$$

We will call the function $\zeta_G^{(r)}$, the generalized ζ function representation of G with r -insertions.

In the case where G is a compact connected simply connected real Lie group, the study of these ζ function with r -insertions, has led to numerous work. First of all the values of these series when s is an even positive integer are connected to the volume of the moduli space of flat connection on a surface with boundaries with prescribed valued of the holonomy in a conjugacy class. See [30] for the statement of the theorem and [24] for a rigorous proof of it. When s is an even positive integer, the study of the function $(\gamma_1, \dots, \gamma_r) \mapsto \zeta^{(r)}(\gamma_1, \dots, \gamma_r)(s)$ is developped in [26], where the relation of these functions with generalization of Bernouilli polynomials is studied. For recent works on this subject see [19][5]. In particular it is shown that if $s \in 2\mathbb{N}$ and if we write $\gamma_j = \exp(v_j)$ with $v_j \in \mathfrak{h}$ where \mathfrak{h} is a fixed Cartan Lie subalgebra of \mathfrak{g} , the function $\mathfrak{h}^r \rightarrow \mathbb{C}, (v_1, \dots, v_r) \mapsto \zeta^{(r)}(e^{v_1}, \dots, e^{v_r})(s)$ is a "locally" polynomial function, meaning that there is a family of open sets (U_i) of \mathfrak{h}^r such that $\cup_{i \in I} \overline{U_i} = \mathfrak{h}^r$, and the restriction of $\zeta^{(r)}(\cdot, s)$ to U_i is a polynomial function.

This is easily seen in the case where $G = SU(2)$: we denote π_n the representation of G of dimension $n+1, n \in \mathbb{N}$. We denote T the Cartan torus of diagonal matrices and let $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have $\chi_{\pi_n}(e^{i\theta t}) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$. Therefore, for s non negative even integer, we have

$$\begin{aligned} \zeta^{(r)}(e^{i\theta_1 t}, \dots, e^{i\theta_r t})(s) &= \sum_{n \in \mathbb{N}} \frac{1}{(n+1)^{s+r}} \prod_{j=1}^r \frac{\sin((n+1)\theta_j)}{\sin(\theta_j)} \\ &= \sum_{m \in \mathbb{N} \setminus \{0\}} \frac{1}{m^{s+r}} \prod_{j=1}^r \frac{e^{im\theta_j} - e^{-im\theta_j}}{2i\sin(\theta_j)} \\ &= \frac{1}{(2i)^r \prod_{j=1}^r \sin(\theta_j)} \sum_{\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}} \prod_{j=1}^r \epsilon_j \sum_{m \in \mathbb{N} \setminus \{0\}} \frac{1}{m^{s+r}} \exp(im \sum_{j=1}^r \epsilon_j \theta_j) \\ &= \frac{1}{2(2i)^r \prod_{j=1}^r \sin(\theta_j)} \sum_{\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}} \prod_{j=1}^r \epsilon_j \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{m^{s+r}} \exp(im \sum_{j=1}^r \epsilon_j \theta_j) \text{ (because } s \in 2\mathbb{N}) \\ &= \frac{1}{2(2i)^r \prod_{j=1}^r \sin(\theta_j)} \sum_{\epsilon_1, \dots, \epsilon_r \in \{\pm 1\}} \left(\prod_{j=1}^r \epsilon_j \right) b_{s+r} \left(\frac{1}{2\pi} \sum_{j=1}^r \epsilon_j \theta_j \right) \end{aligned}$$

where b_k is the function defined by $b_k(x) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{e^{2i\pi mx}}{m^k}$, related to the k -th Bernoulli polynomial B_k by $-\frac{k!}{(2i\pi)^k} b_k(x) = B_k(x - \lfloor x \rfloor)$ with $k \geq 2$.

In the case where G is a compact p -adic Lie group, much less are known on these zeta functions with insertions. When $p = 1$, it is shown (theorem 1.2) in [14] that for Fab uniform pro- p group, for fixed $\gamma \in G$, the function $s \mapsto \zeta^{(1)}(\gamma, s)$ is a rational function in p^s . A question that is pending is the dependence on $\gamma_1, \dots, \gamma_r$. There is no result in the litterature on this question. We will answer this question for some selected examples: in the present article for the case $GL(2, \mathbb{F}_q), PGL(2, \mathbb{F}_q)$ (section 2) and in [4] for the case $GL(2, \mathbb{Z}_p), PGL(2, \mathbb{Z}_p)$.

2.3. Generalized zeta function and non orientable surfaces: Schur-Frobenius formula. One generalisation that can be considered is the case where Σ is a compact non orientable surface. In this case, Σ is homeomorphic to the connected sum of g copies of $\mathbb{P}^2(\mathbb{R})$ with $g \geq 1$ and the Euler characteristic of Σ is $\chi(\Sigma) = 2 - g$.

We fix a point x on Σ , and consider the set of group morphisms $X_G(x, \Sigma) = Hom(\pi_1(x, \Sigma), G)$. $X_G(x, \Sigma)/AdG$ is in bijection with the set of G -principal bundle over Σ .

A generalisation of Mednykh formula gives [11][25]:

$$(13) \quad \frac{|X_G(x, \Sigma)|}{|G|} = |G|^{-\chi(\Sigma)} \sum_{\pi \in \widehat{G}, \nu_2(\pi) \neq 0} (\nu_2(\pi) \dim \pi)^{\chi(\Sigma)},$$

where $\nu_2(\pi) \in \{+1, 0, -1\}$ is the Frobenius-Schur indicator of π and the sum is restricted to the value of π such that $\nu_2(\pi)$ is different of zero. The Frobenius indicator is defined by the formula

$$(14) \quad \nu_2(\pi) = \int_G \chi_\pi(g^2) dg.$$

In fact Frobenius and Schur already obtained this formula in 1906 while studying the number of solutions (R_1, \dots, R_g) in G^g of the equation $R_1^2 \cdots R_g^2 = e$.

Proof: This generalisation is easily obtained using the fact that $\pi_1(\Sigma, x)$ admits a presentation by generators r_1, \dots, r_g with relation $r_1^2 \cdots r_g^2 = e$. We have $X_G(x, \Sigma) = \{(R_1, \dots, R_g) \in G^g, R_1^2 \cdots R_g^2 = e\}$. If we introduce the function $\theta_{\mathbb{P}^2(\mathbb{R})} : G \rightarrow \mathbb{N}, \theta_{\mathbb{P}^2(\mathbb{R})}(g) = |\{h \in G, h^2 = g\}|$, it admits the following Fourier decomposition

$$(15) \quad \theta_{\mathbb{P}^2(\mathbb{R})} = \sum_{\pi \in \widehat{G}} \nu_2(\pi) \chi_\pi.$$

From

$$\begin{aligned} |X_G(x, \Sigma)| &= |G|^g \int_{G^{2g}} \theta_{\mathbb{P}^2(\mathbb{R})}(k_1) \cdots \theta_{\mathbb{P}^2(\mathbb{R})}(k_g) \delta(k_1 \dots k_g) dk_1 \cdots dk_g \\ &= |G|^{g-1} \underbrace{(\theta_{\mathbb{P}^2(\mathbb{R})} * \cdots * \theta_{\mathbb{P}^2(\mathbb{R})})}_{g \text{ times}}(e) \end{aligned}$$

we obtain the generalization of Mednikh formula. This is the Schur-Frobenius proof. \square

Therefore we can define for any rigid compact topological group G , the following functions

$$(16) \quad \zeta_{G,\epsilon}(s) = \sum_{\pi \in \hat{G}, \nu_2(\pi) = \epsilon} \frac{1}{(\dim \pi)^s}, s \in \mathbb{C}, \epsilon \in \{+1, -1, 0\}.$$

We have $\zeta_G = \zeta_{G,+1} + \zeta_{G,0} + \zeta_{G,-1}$ and $\frac{|X_G(x, \Sigma)|}{|G|} = |G|^{-\chi(\Sigma)} (\zeta_{G,+1}(-\chi(\Sigma)) + (-1)^{\chi(\Sigma)} \zeta_{G,-1}(-\chi(\Sigma)))$.

The value of $\zeta_{G,+1}(n) + (-1)^n \zeta_{G,-1}(n)$, for n positive integer, has been considered in [30][13] where G is a connected simply connected semisimple compact Lie group, in connection with the volume of the moduli space of G flat connections on Σ , where a relation such as (9) holds true.

Mednikh formula has also a generalisation to the case of compact non orientable surface with boundaries.

Let G be a finite group, the ζ_G function can be generalized to the case where one consider connected compact non-orientable topological surfaces Σ with boundaries. Let Σ be a connected compact non-orientable topological surface Σ of Euler characteristic $\chi(\Sigma) = 2 - g$, denote $\Sigma' = \Sigma \setminus (\cup_{j=1}^r D_j)$ where D_j are open disks included in Σ whose closures do not intersect pairwise. We have $\partial \Sigma' = C_1 \cup \cdots \cup C_r$, where C_1, \dots, C_r are homeomorphic to circles. Let x be an interior point of Σ' , and fix closed curves ℓ_1, \dots, ℓ_r originating from x such that the homotopy class of ℓ_j is the same as C_j . The homotopy group $\pi_1(\Sigma', x)$ is isomorphic to the group generated by $r_i, i = 1, \dots, g, m_j, j = 1, \dots, r$, with relation $\prod_{i=1}^g r_i^2 \prod_{j=1}^r m_j = e$.

We fix $\gamma_1, \dots, \gamma_r \in G$, we can still define the finite set $X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)$ of principal G -bundles over Σ' such that the holonomy along ℓ_j belongs to the conjugacy class of $\gamma_j, j = 1 \cdots r$.

Proposition 1.

(17)

$$\frac{|X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)|}{|G|} = |G|^{-\chi(\Sigma)} \prod_{j=1}^r |O(\gamma_j)| \sum_{\pi \in \hat{G}, \nu_2(\pi) \neq 0} \frac{1}{(\nu_2(\pi) \dim(\pi))^{-\chi(\Sigma)}} \frac{\prod_{j=1}^r \chi_\pi(\gamma_j)}{(\dim \pi)^r}.$$

Proof: Direct computation, using

$$|X_G(x, \Sigma, \gamma_1, \dots, \gamma_r)| = |G|^{g+r} \int_{G^{g+r}} \delta\left(\prod_{i=1}^g R_i^2 \prod_{j=1}^r M_j\right) \prod_{j=1}^r \delta_{O(\gamma_j)}(M_j) \prod_i dR_i \prod_j dM_j.$$

□.

We are therefore led to define, for a rigid compact group G and $\epsilon \in \{+1, -1, 0\}$ a generalized zeta functions $\zeta_{G,\epsilon}^{(r)} : G^r \times \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$(18) \quad \zeta_{G,\epsilon}^{(r)}(\gamma_1, \dots, \gamma_r)(s) = \sum_{\pi \in \widehat{G}, \nu_2(\pi) = \epsilon} \frac{\prod_{j=1}^r \chi_\pi(\gamma_j)}{(\dim \pi)^{s+r}}, s \in \mathbb{C}, \gamma_1, \dots, \gamma_r \in G.$$

To our knowledge these functions do not appear in the litterature and their properties are completely unknown.

2.4. Generalization of Mednikh formulas and the ζ function representation of the double $D(G)$. Let G be a finite group, Σ a compact orientable topological surface of genus g , we can express the number $|Hom(\pi_1(\Sigma), G)/AdG|$ in term of evaluation at $2g-2$ of the ζ representation function of the quantum double of G . This formula appears in [7] and has been proved using topological field theory. We give here a simple proof of it, answering a question asked in [7], which admit straightforward generalisation to the non orientable case and to the case with boundaries that we establish.

Choose a section $\iota : G/AdG \rightarrow G$, and denote $C_\iota(O) = C(\iota(O))$ where $C(k), k \in G$ denotes the centralizer in G of the element k .

The double $D(G)$ of a finite group is a quasitriangular Hopf \mathbb{C} -algebra which is the quantum double of the algebra of complex functions on G . This algebra was introduced and its representations were classified in [8]. The aim of this work was to understand the work [9] from the point of view of braided tensor categories. It is shown that for each couple (O, ρ) where O is a conjugacy class of G and ρ is an irreducible finite dimensional representation of $C_\iota(O)$ there exists an irreducible finite dimensional representation of $D(G)$ denoted $\Pi(O, \rho)$ of dimension $|O|\dim \rho$. We denote $\widehat{D(G)}$ the set of isomorphism classes of irreducible representations of $D(G)$. The set $\{\Pi(O, \rho)\}$ are representatives of $\widehat{D(G)}$ and this is independent of the choice of ι .

We can therefore define the function $\zeta_{D(G)}$

$$(19) \quad \zeta_{D(G)}(s) = \sum_{\Pi \in \widehat{D(G)}} \frac{1}{(\dim \Pi)^s}, s \in \mathbb{C}$$

$$(20) \quad = \sum_{O \in G/AdG, \rho \in \widehat{C_\iota(O)}} \frac{1}{(|O|\dim \rho)^s}.$$

Proposition 2. *The generalization of Mednikh formula holds:*

$$\begin{aligned}
|Hom(\pi_1(\Sigma), G)/AdG| &= \sum_{k \in G} \frac{|C(k)|}{|G|} |C(k)|^{2g-2} \zeta_{C(k)}(2g-2) \\
&= \sum_{O \in G/AdG} |C_\iota(O)|^{2g-2} \zeta_{C_\iota(O)}(2g-2) \\
&= |G|^{2g-2} \zeta_{D(G)}(2g-2).
\end{aligned}$$

Proof: G acts on $X_G(x, \Sigma)$ by adjoint action, applying Burnside counting lemma, we obtain

$$|X_G(x, \Sigma)/AdG| = \frac{1}{|G|} \sum_{k \in G} |X_G(x, \Sigma)^k|$$

where $X_G(x, \Sigma)^k$ is the subset of elements of $X_G(x, \Sigma)$ fixed by k . From $Hom(., G)^k = Hom(., C(k))$, we obtain

$$\begin{aligned}
|X_G(x, \Sigma)/AdG| &= \frac{1}{|G|} \sum_{k \in G} |X_{C(k)}(x, \Sigma)| \\
&= \sum_{k \in G} \frac{|C(k)|}{|G|} |C(k)|^{2g-2} \zeta_{C(k)}(2g-2).
\end{aligned}$$

The second equality of the proposition is a consequence of:

$$\begin{aligned}
&\sum_{k \in G} \frac{|C(k)|}{|G|} |C(k)|^{2g-2} \zeta_{C(k)}(2g-2) = \\
&= \sum_{k \in G} \frac{1}{|O(k)|} \sum_{\rho \in \widehat{C(k)}} |C(k)|^{2g-2} \frac{1}{(dim \rho)^{2g-2}} \\
&= \sum_{O \in G/AdG} \sum_{\rho \in \widehat{C_\iota(O)}} |C_\iota(O)|^{2g-2} \frac{1}{(dim \rho)^{2g-2}} \\
&= \sum_{O \in G/AdG} \sum_{\rho \in \widehat{C_\iota(O)}} |G|^{2g-2} \frac{1}{(|O| dim \rho)^{2g-2}} \\
&= |G|^{2g-2} \zeta_{D(G)}(2g-2).
\end{aligned}$$

□

These formulas can be generalized to the case of surface with boundaries and non orientable surfaces.

We begin with a lemma.

Lemma 1. *Let H be a subgroup of G and $\gamma \in G$, the restriction to H of the characteristic function of the conjugacy class $O(\gamma)$ of G admits the following Fourier transform:*

$$\delta_{O(\gamma)}|_H = \frac{1}{|C(\gamma)|} \sum_{\rho \in \widehat{H}} \chi_\rho Tr(Ind_H^G(\rho)(\gamma^{-1})).$$

Proof: We use the classical notation of group theory: if ϕ is a function on H we extend it on G as ϕ^0 with the definition $\phi^0(x) = \phi(x)$ if $x \in H$ and zero otherwise. The Fourier expansion of the class function $\delta_{O(\gamma)}|_H$ is $\sum_{\rho \in \widehat{H}} a_\rho \chi_\rho$, therefore:

$$\begin{aligned} \bar{a}_\rho &= \frac{1}{|H|} \sum_{h \in H} \delta_{O(\gamma)}(h) \chi_\rho(h) \\ &= \frac{1}{|H|} \sum_{h \in H \cap O(\gamma)} \chi_\rho(h) \\ &= \frac{1}{|H|} \sum_{h \in O(\gamma)} \chi_\rho^0(h) \\ &= \frac{1}{|H||C(\gamma)|} \sum_{x \in G} \chi_\rho^0(x\gamma x^{-1}) \\ &= \frac{1}{|C(\gamma)|} \text{Tr}(\text{Ind}_H^G(\rho)(\gamma)). \end{aligned}$$

□

Proposition 3. *The generalization of Mednikh formula holds:*

$$\begin{aligned} &|X_G(\Sigma, \gamma_1, \dots, \gamma_r)/\text{Ad}G| = \\ &= \frac{\prod_{j=1}^r |O(\gamma_j)|}{|G|^{r+1}} \sum_{k \in G} |C(k)|^{2g+r-1} \sum_{\rho \in \widehat{C(k)}} \frac{\prod_{j=1}^r \text{Tr}(\text{Ind}_{C(k)}^G(\rho)(\gamma_j))}{(\dim \rho)^{2g-2+r}} \end{aligned}$$

Proof: Straightforward □.

In the case where Σ is a compact non orientable surface of Euler characteristic $2 - g$ we have:

Proposition 4.

$$|Hom(\pi(\Sigma), G)/\text{Ad}G| = \sum_{O \in G/\text{Ad}G} |C_\iota(O)|^{-\chi(\Sigma)} \sum_{\rho \in \widehat{C_\iota(O)}, \nu_2(\rho) \neq 0} (\nu_2(\rho) \dim \rho)^{\chi(\Sigma)},$$

Proof: Straightforward. □

In the case where Σ is a compact non orientable surface of Euler characteristic $2 - g$, with boundaries we have:

Proposition 5.

$$\begin{aligned} &|X_G(\Sigma, \gamma_1, \dots, \gamma_r)/\text{Ad}G| = \\ &= \frac{\prod_{j=1}^r |O(\gamma_j)|}{|G|^{r+1}} \sum_{k \in G} |C(k)|^{g+r-1} \sum_{\rho \in \widehat{C(k)}, \nu_2(\rho) \neq 0} \frac{\prod_{j=1}^r \text{Tr}(\text{Ind}_{C(k)}^G(\rho)(\gamma_j))}{(\nu_2(\rho) \dim \rho)^{-\chi(\Sigma)} (\dim \rho)^r} \end{aligned}$$

Proof: Straightforward. □.

We will now explicitly compute the functions defined in this section in two simple cases $GL(2, \mathbb{F}_q)$ and $PGL(2, \mathbb{F}_q)$.

3. EXPLICIT COMPUTATION FOR THE GROUPS $GL(2, \mathbb{F}_q)$, $PGL(2, \mathbb{F}_q)$

3.1. Computation of generalized ζ functions in the $GL(2, \mathbb{F}_q)$ case.

We have collected in the appendix the description of conjugacy classes, irreducible representations of these groups and values of the Frobenius-Schur indicator. We use the notation $\mathbb{F} = \mathbb{F}_q$ and $\mathbb{E} = \mathbb{F}_{q^2}$.

From the classification of irreducible representations of $GL(2, \mathbb{F}_q)$, the zeta function is straightforwardly computed and reads:

$$(21) \quad \zeta_{GL(2, \mathbb{F}_q)}(s) = q - 1 + \frac{(q-1)(q-2)}{2(q+1)^s} + \frac{q-1}{q^s} + \frac{(q-1)q}{2(q-1)^s}.$$

Let $\gamma_1, \dots, \gamma_r$ elements of $G = GL(2, \mathbb{F}_q)$, we consider $\zeta_G^{(r)}(\gamma_1, \dots, \gamma_r)(s)$. Note that the group of characters of G acts by tensorisation on the set of irreducible representations, therefore we have, for any $\mu \in \widehat{\mathbb{F}_q^\times}$

$$(22) \quad \zeta_G^{(r)}(\gamma_1, \dots, \gamma_r)(s) = \chi_\mu(\gamma_1 \cdots \gamma_r) \zeta_G^{(r)}(\gamma_1, \dots, \gamma_r)(s).$$

This implies that the function $\zeta_G^{(r)}(\gamma_1, \dots, \gamma_r)(s)$ is null if $\det(\gamma_1 \cdots \gamma_r) \neq 1$.

We will compute the generalized zeta functions $\zeta^{(1)}$ for the 4 types of conjugacy classes.

These can easily be done using the identities valid for any $x \in \mathbb{F}^\times$

$$\begin{aligned} \sum_{\mu \in \widehat{\mathbb{F}^\times}} \mu(x) &= (q-1)\delta(x), \\ \sum_{\substack{\{\mu_1, \mu_2\} \subset \widehat{\mathbb{F}^\times} \\ \mu_1 \neq \mu_2}} \mu_1(x)\mu_2(y) &= \frac{1}{2} \left(\sum_{\mu \in \widehat{\mathbb{F}^\times}} \mu(x)^2 \right) - \frac{1}{2} \sum_{\mu \in \widehat{\mathbb{F}^\times}} \mu(x)^2 = \frac{1}{2}(q-1)^2\delta(x) - \frac{1}{2}(q-1)\delta(x^2), \\ \sum_{\nu \in (\widehat{\mathbb{E}^\times} \setminus \widehat{\mathbb{F}^\times})/\sim} \nu(x) &= \frac{1}{2} \sum_{\nu \in \widehat{\mathbb{E}^\times}} \nu(x) - \frac{1}{2} \sum_{\mu \in \widehat{\mathbb{F}^\times}} \mu(N(x)) = \frac{1}{2}(q^2-1)\delta(x) - \frac{1}{2}(q-1)\delta(x^2). \end{aligned}$$

We therefore have

$$\begin{aligned} \zeta^{(1)}(c_1(x))(s) &= \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(x)^2}{1^{s+1}} + \sum_{\substack{\{\mu_1, \mu_2\} \subset \widehat{\mathbb{F}^\times} \\ \mu_1 \neq \mu_2}} \frac{(q+1)\mu_1(x)\mu_2(x)}{(q+1)^{s+1}} + \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{q\mu(x)^2}{q^{s+1}} + \sum_{\nu \in (\widehat{\mathbb{E}^\times} \setminus \widehat{\mathbb{F}^\times})/\sim} \frac{(q-1)\nu(x)}{(q-1)^{s+1}} \\ &= (q-1)\delta(x^2) + \frac{1}{2}(q+1)^{-s}((q-1)^2\delta(x) - (q-1)\delta(x^2)) + \\ &\quad q^{-s}(q-1)\delta(x^2) + \frac{1}{2}(q-1)^{-s}((q^2-1)\delta(x) - (q-1)\delta(x^2)). \end{aligned}$$

$$\begin{aligned} \zeta^{(1)}(c_2(x))(s) &= \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(x)^2}{1^{s+1}} + \sum_{\substack{\{\mu_1, \mu_2\} \subset \widehat{\mathbb{F}^\times} \\ \mu_1 \neq \mu_2}} \frac{\mu_1(x)\mu_2(x)}{(q+1)^{s+1}} + \sum_{\nu \in (\widehat{\mathbb{E}^\times} \setminus \widehat{\mathbb{F}^\times})/\sim} \frac{-\nu(x)}{(q-1)^{s+1}} \\ &= (q-1)\delta(x^2) + \frac{1}{2}(q+1)^{-s-1}((q-1)^2\delta(x) - (q-1)\delta(x^2)) - \frac{1}{2}(q-1)^{-s-1}((q^2-1)\delta(x) - (q-1)\delta(x^2)). \end{aligned}$$

$$\begin{aligned}\zeta^{(1)}(c_3(x, y))(s) &= \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(x)\mu(y)}{1^{s+1}} + \sum_{\substack{\{\mu_1, \mu_2\} \subset \widehat{\mathbb{F}^\times} \\ \mu_1 \neq \mu_2}} \frac{\mu_1(x)\mu_2(y) + \mu_1(y)\mu_2(x)}{(q+1)^{s+1}} + \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(xy)}{q^{s+1}} \\ &= (q-1)\delta(xy) + (q+1)^{-s-1}((q-1)^2\delta(x)\delta(y) - (q-1)\delta(xy)) + q^{-s-1}(q-1)\delta(xy).\end{aligned}$$

$$\begin{aligned}\zeta^{(1)}(c_4(\lambda))(s) &= \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(\lambda\bar{\lambda})}{1^{s+1}} - \sum_{\mu \in \widehat{\mathbb{F}^\times}} \frac{\mu(\lambda\bar{\lambda})}{q^{s+1}} - \sum_{\nu \in (\widehat{\mathbb{F}^\times} \setminus \widehat{\mathbb{F}^\times})/\sim} \frac{\nu(\lambda) + \nu(\bar{\lambda})}{(q-1)^{s+1}} \\ &= (q-1)\delta(\lambda\bar{\lambda}) - (q-1)q^{-s-1}\delta(\lambda\bar{\lambda}) - (q^2-1)(q-1)^{-s-1}\delta(\lambda) + (q-1)(q-1)^{-s-1}\delta(\lambda\bar{\lambda}).\end{aligned}$$

$\zeta^{(1)}(\gamma, s)$ vanishes when the determinant of γ is not equal to 1, this is in sharp contrast with the $PGL(2, \mathbb{F}_q)$ case that we will consider in the next section.

We can compute explicitly the zeta function of the double $D(G)$.

We first describe the centralizer of elements of G .

- $C(c_1(x)) = GL(2, \mathbb{F})$.
- $C(c_2(x)) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a \in \mathbb{F}^\times, b \in \mathbb{F} \right\} \simeq \mathbb{F}^\times \times \mathbb{F}$ abelian group of order $q(q-1)$.
- $C(c_3(x, y)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, (a, b) \in (\mathbb{F}^\times)^2 \right\} \simeq (\mathbb{F}^\times)^2$ abelian group of order $(q-1)^2$.
- $C(c_4(\lambda)) = \left\{ \begin{pmatrix} a & -c\lambda\bar{\lambda} \\ c & a + c(\lambda + \bar{\lambda}) \end{pmatrix}, (a, b) \in \mathbb{F}^2 \setminus \{(0, 0)\} \right\}$ abelian group of order $q^2 - 1$.

Therefore we have

$$\begin{aligned}\zeta_{D(GL(2, \mathbb{F}))}(s) &= \sum_{x \in \mathbb{F}^\times} \zeta_{C(c_1(x))}(s) + \sum_{x \in \mathbb{F}^\times} \frac{1}{|C(c_2(x))|^s} |\mathbb{F}^\times \times \mathbb{F}| + \\ &+ \sum_{\substack{\{x, y\} \subset \mathbb{F}^\times \\ x \neq y}} \frac{1}{|C(c_3(x, y))|^s} |\mathbb{F}^\times \times \mathbb{F}^\times| + \sum_{\substack{\lambda \in \mathbb{F} \setminus \mathbb{F}^\times \\ \lambda \sim \bar{\lambda}}} \frac{1}{|C(c_4(\lambda))|^s} |\mathbb{F}^2 \setminus \{(0, 0)\}|. \\ \zeta_{D(GL(2, \mathbb{F}))}(s) &= (q-1)^2 + \frac{(q-1)^2(q-2)}{2(q+1)^s} + \frac{(q-1)^2}{q^s} + \frac{(q-1)^2q}{2(q-1)^s} + \\ &+ \frac{q(q-1)^2}{(q(q-1))^s} + \frac{(q-1)^3(q-2)}{2(q-1)^{2s}} + \frac{q(q-1)^2(q+1)}{2(q^2-1)^s}.\end{aligned}$$

We now give the form of $\zeta_{GL(2, \mathbb{F}_q), \epsilon}$ functions.

There is no Frobenius Schur indicator having -1 value, therefore $\zeta_{GL(2, \mathbb{F}), -1} = 0$.

When q is odd we have

$$\zeta_{GL(2, \mathbb{F}), 1}(s) = 2 + \frac{q-1}{2(q+1)^s} + \frac{2}{q^s} + \frac{q-1}{2(q-1)^s},$$

when q is even we have

$$\zeta_{GL(2, \mathbb{F}), 1}(s) = 1 + \frac{q-2}{2(q+1)^s} + \frac{1}{q^s} + \frac{q}{2(q-1)^s}.$$

The expression of $\zeta_{GL(2, \mathbb{F}), 0}$ is obtained from $\zeta_{GL(2, \mathbb{F}), 0} = \zeta_{GL(2, \mathbb{F})} - \zeta_{GL(2, \mathbb{F}), 1}$.

3.2. Computation of generalized ζ functions in the $PGL(2, \mathbb{F}_q)$ case.

From the structure classification of irreducible representations of $PGL(2, \mathbb{F}_q)$, the zeta function is straightforwardly computed and reads:

- In odd characteristic one has: $\zeta_{PGL(2, \mathbb{F}_q)}(s) = 2 + \frac{(q-3)}{2(q+1)^s} + \frac{2}{q^s} + \frac{(q-1)}{2(q-1)^s}$
- in characteristic 2 one has: $\zeta_{PGL(2, \mathbb{F}_q)}(s) = 1 + \frac{(q-2)}{2(q+1)^s} + \frac{1}{q^s} + \frac{q}{2(q-1)^s}$.

From the property of the Schur-Frobenius indicator of irreducible representations of $PGL(2, \mathbb{F})$, we have:

$$\zeta_{PGL(2, \mathbb{F}), -1} = \zeta_{PGL(2, \mathbb{F}), 0} = 0.$$

We now give explicit expressions of zeta functions with insertions.

We denote by \mathcal{M} the set of equivalence classes for the relation $\mu \sim \mu^{-1}$ on the set of characters $\mu \in \widehat{\mathbb{F}^\times}$ such that $\mu \neq 1$ in the case q even and $\mu \notin \{1, \epsilon\}$ in the case q odd. This set is in bijection with the set of inequivalent representations of $PGL(2, \mathbb{F})$ of principal type.

We denote by \mathcal{N} the set of equivalence classes for the relation $\nu \sim \nu^{-1}$ on the set of primitive characters $\nu \in \widehat{\mathbb{E}^\times} \setminus \widehat{\mathbb{F}^\times}$ such that $\nu|_{\mathbb{F}^\times} = id$. This set is in bijection with the set of inequivalent representations of $PGL(2, \mathbb{F})$ of cuspidal type.

- We first assume that q is odd.

Computation of the ζ function with one insertion.

$$\zeta^{(1)}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, s\right) = 2 + \frac{(q-3)}{2(q+1)^{s+1}} - \frac{(q-1)}{2q^{s+1}}.$$

Let $x \in \mathbb{F}^\times, x \neq 1$,

$$\begin{aligned} \zeta^{(1)}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, s\right) &= 1 + \epsilon(x) + \sum_{\mu \in \mathcal{M}} \frac{\mu(x) + \mu(x)^{-1}}{(q+1)^{s+1}} + \frac{1 + \epsilon(x)}{q^{s+1}} \\ &= 1 + \epsilon(x) + \frac{1}{(q+1)^{s+1}}((q-1)\delta(x) - 1 - \epsilon(x)) + \frac{1 + \epsilon(x)}{q^{s+1}} \\ &= \left(1 + \frac{1}{q^{s+1}} - \frac{1}{(q+1)^{s+1}}\right)(1 + \epsilon(x)). \end{aligned}$$

Let $\epsilon_{\mathbb{E}} : \mathbb{E}^\times \rightarrow \{-1, 1\}$ be the morphism of multiplicative group defined by $\epsilon_{\mathbb{E}}(\lambda) = -1$ if and only if λ is not a square in \mathbb{E}^\times .

Let $\lambda \in \mathbb{E} \setminus \mathbb{F}$,

$$\begin{aligned} \zeta^{(1)}\left(\begin{pmatrix} 0 & -\lambda\bar{\lambda} \\ 1 & \lambda + \bar{\lambda} \end{pmatrix}, s\right) &= \\ &= 1 + \epsilon(\lambda\bar{\lambda}) - \sum_{\nu \in \mathcal{N}} \frac{\nu(\lambda) + \nu(\bar{\lambda})}{(q-1)^{s+1}} - \frac{1 + \epsilon(\lambda\bar{\lambda})}{q^{s+1}} \\ &= \left(1 + \frac{1}{(q-1)^{s+1}} - \frac{1}{q^{s+1}}\right) + \epsilon(\lambda\bar{\lambda})\left(1 - \frac{1}{q^{s+1}}\right) + \frac{1}{(q-1)^{s+1}}\epsilon_{\mathbb{E}}(\lambda). \end{aligned}$$

where we have used the identity $\sum_{\nu \in \mathcal{N}} (\nu(\lambda) + \nu(\bar{\lambda})) = -1 - \epsilon_{\mathbb{E}}(\lambda)$, for $\lambda \in \mathbb{E} \setminus \mathbb{F}$.

Computation of the generalized function with r-insertions in the case where all the conjugacy classes are diagonal.

Let $\gamma_i = \begin{pmatrix} x_i & 0 \\ 0 & 1 \end{pmatrix}$, $i = 1, \dots, r$ with $x_i \in \mathbb{F}^\times \setminus \{1\}$, we have

$$\begin{aligned} \zeta^{(r)}(\gamma_1, \dots, \gamma_r, s) &= 1 + \epsilon(x_1 \cdots x_r) + \sum_{\mu \in \mathcal{M}} \frac{(\mu(x_1) + \mu(x_1)^{-1}) \cdots (\mu(x_r) + \mu(x_r)^{-1})}{(q+1)^{s+r}} + \\ &\quad + \frac{1 + \epsilon(x_1 \cdots x_r)}{q^{s+r}} \\ &= (1 + \epsilon(x_1 \cdots x_r))\left(1 + \frac{1}{q^{s+r}}\right) + \frac{1}{2(q+1)^{s+r}} \sum_{\substack{\eta_i \in \{\pm 1\} \\ i=1, \dots, r}} ((q-1)\delta(x_1^{\eta_1} \cdots x_r^{\eta_r}) - 1 - \epsilon(x_1^{\eta_1} \cdots x_r^{\eta_r})). \end{aligned}$$

Computation of the generalized function with r-insertions in the case where all the conjugacy classes are elliptic.

Let $\gamma_i = \begin{pmatrix} 0 & -\lambda_i\bar{\lambda}_i \\ 1 & \lambda_i + \bar{\lambda}_i \end{pmatrix}$, $i = 1, \dots, r$ with $\lambda_i \in \mathbb{E}^\times \setminus \mathbb{F}$, we have

$$\begin{aligned} \zeta^{(r)}(\gamma_1, \dots, \gamma_r, s) &= 1 + \epsilon(\lambda_1\bar{\lambda}_1 \cdots \lambda_r\bar{\lambda}_r) + (-1)^r \sum_{\nu \in \mathcal{N}} \frac{(\nu(\lambda_1) + \nu(\bar{\lambda}_1)) \cdots (\nu(\lambda_r) + \nu(\bar{\lambda}_r))}{(q-1)^{s+r}} + (-1)^r \frac{1}{q^{s+r}} \\ &= (1 + (-1)^r \frac{1}{q^{s+r}}) + \frac{(-1)^r}{2(q-1)^{s+r}} \sum_{\substack{\eta_i \in \{id, \sigma\} \\ i=1, \dots, r}} ((q+1)\phi_{\mathbb{F}}(\eta_1(\lambda_1) \cdots \eta_r(\lambda_r)) - 1 - \epsilon_{\mathbb{E}}(\eta_1(\lambda_1) \cdots \eta_r(\lambda_r))), \end{aligned}$$

where $\phi_{\mathbb{F}} : \mathbb{E} \rightarrow \{0, 1\}$ is the characteristic function of \mathbb{F} which satisfies $\sum_{\nu \in \mathcal{N}} (\nu(\lambda) + \nu(\bar{\lambda})) = (q+1)\phi_{\mathbb{F}}(\lambda) - 1 - \epsilon_{\mathbb{E}}(\lambda)$ for $\lambda \in \mathbb{E}^\times$.

Computation of the generalized function with r insertions in the mixed case where m are diagonal and n are elliptic, with $m \geq 1$ and $n \geq 1$.

We denote $\gamma_i = \begin{pmatrix} x_i & 0 \\ 0 & 1 \end{pmatrix}$, $i = 1, \dots, m$, with $x_i \in \mathbb{F}^\times \setminus \{1\}$, $\gamma_{i+m} = \begin{pmatrix} 0 & -\lambda_i\bar{\lambda}_i \\ 1 & \lambda_i + \bar{\lambda}_i \end{pmatrix}$, $i = 1, \dots, n$ with $\lambda_i \in \mathbb{E}^\times \setminus \mathbb{F}$, the structure of the character

table implies that

$$\zeta^{(r)}(\gamma_1, \dots, \gamma_{m+n}, s) = (1 + \frac{(-1)^n}{q^{s+r}})(1 + \epsilon(x_1 \dots x_m \lambda_1 \overline{\lambda_1} \dots \lambda_n \overline{\lambda_n})).$$

- We now assume q is a power of 2.

Computation of the ζ function with one insertion.

$$\zeta^{(1)}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, s\right) = 1 + \frac{(q-2)}{2(q+1)^{s+1}} - \frac{(q)}{2q^{s+1}}.$$

Let $x \in \mathbb{F}^\times, x \neq 1$,

$$\begin{aligned} \zeta^{(1)}\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, s\right) &= 1 + \sum_{\mu \in \mathcal{M}} \frac{\mu(x) + \mu(x)^{-1}}{(q+1)^{s+1}} + \frac{1}{q^{s+1}} \\ &= 1 + \frac{1}{(q+1)^{s+1}}((q-1)\delta(x) - 1) + \frac{1}{q^{s+1}} = 1 - \frac{1}{(q+1)^{s+1}} + \frac{1}{q^{s+1}}. \end{aligned}$$

Let $\lambda \in \mathbb{E} \setminus \mathbb{F}$,

$$\begin{aligned} \zeta^{(1)}\left(\begin{pmatrix} 0 & -\lambda\overline{\lambda} \\ 1 & \lambda + \overline{\lambda} \end{pmatrix}, s\right) &= 1 - \sum_{\nu \in \mathcal{N}} \frac{\nu(\lambda) + \nu(\overline{\lambda})}{(q-1)^{s+1}} - \frac{1}{q^{s+1}} \\ &= 1 - \frac{1}{q^{s+1}} + \frac{1}{(q-1)^{s+1}}, \end{aligned}$$

where we have used the identity $\sum_{\nu \in \mathcal{N}} (\nu(\lambda) + \nu(\overline{\lambda})) = -1$ for $\lambda \in \mathbb{E} \setminus \mathbb{F}$.

Computation of the generalized function with r insertions in the case where all the conjugacy classes are diagonal.

Let $\gamma_i = \begin{pmatrix} x_i & 0 \\ 0 & 1 \end{pmatrix}, i = 1, \dots, r$ with $x_i \in \mathbb{F}^\times \setminus \{1\}$, we have

$$\begin{aligned} \zeta^{(r)}(\gamma_1, \dots, \gamma_r, s) &= 1 + \sum_{\mu \in \mathcal{M}} \frac{(\mu(x_1) + \mu(x_1)^{-1}) \cdots (\mu(x_r) + \mu(x_r)^{-1})}{(q+1)^{s+r}} + \frac{1}{q^{s+r}} \\ &= (1 + \frac{1}{q^{s+r}}) + \frac{1}{2(q+1)^{s+r}} \sum_{\eta_i = \pm 1} ((q-1)\delta(x_1^{\eta_1} \cdots x_r^{\eta_r}) - 1). \end{aligned}$$

Computation of the generalized function with r insertions in the case where all the conjugacy classes are elliptic.

Let $\gamma_i = \begin{pmatrix} 0 & -\lambda_i \overline{\lambda_i} \\ 1 & \lambda_i + \overline{\lambda_i} \end{pmatrix}, i = 1, \dots, r$ with $\lambda_i \in \mathbb{E}^\times \setminus \mathbb{F}$, we have

$$\begin{aligned} \zeta^{(r)}(g_1, \dots, g_r, s) &= 1 + (-1)^r \sum_{\nu \in \mathcal{N}} \frac{(\nu(\lambda_1) + \nu(\overline{\lambda_1})) \cdots (\nu(\lambda_r) + \nu(\overline{\lambda_r}))}{(q-1)^{s+r}} \frac{1}{q^{s+r}} \\ &= (1 - \frac{1}{q^{s+r}}) + \frac{(-1)^{r+1}}{2(q-1)^{s+r}} \sum_{\eta_i \in \{id, \sigma\}} ((q+1)\phi_{\mathbb{F}}(\eta_1(\lambda_1) \cdots \eta_r(\lambda_r)) - 1), \end{aligned}$$

where $\phi_{\mathbb{F}} : \mathbb{E} \rightarrow \{0, 1\}$ is the characteristic function of \mathbb{F} and satisfies $\sum_{\nu \in \mathcal{N}} (\nu(\lambda) + \nu(\overline{\lambda})) = (q+1)\phi_{\mathbb{F}}(\lambda) - 1$ for $\lambda \in \mathbb{E}^\times$.

Computation of the generalized function with r insertions in the mixed case where m are diagonal and n are elliptic, with $m \geq 1$ and $n \geq 1$.

We denote $\gamma_i = \begin{pmatrix} x_i & 0 \\ 0 & 1 \end{pmatrix}, i = 1, \dots, m$, with $x_i \in \mathbb{F}^\times \setminus \{1\}$, $\gamma_{i+m} = \begin{pmatrix} 0 & -\lambda_i \overline{\lambda_i} \\ 1 & \lambda_i + \overline{\lambda_i} \end{pmatrix}, i = 1, \dots, n$ with $\lambda_i \in \mathbb{E}^\times \setminus \mathbb{F}$, the structure of the character table implies that

$$\zeta^{(r)}(\gamma_1, \dots, \gamma_r, s) = 1 + (-1)^n \frac{1}{q^{s+r}}.$$

4. CONCLUSION

This article is giving a review of Mednikh formulas and its generalisations and is pushing forward the definition of generalized ζ representation functions of groups. These generalized functions have not received up to now any attention in the case of p -adic Lie groups, although they are under active study in the case of compact real groups.

The results obtained in this article, although very elementary, are the first necessary step to understand generalized zeta functions with insertions for groups such as $GL(2, \mathcal{O})$ and $PGL(2, \mathcal{O})$ which will be developed in [4]. There are different lines of investigation that can be pursued.

It would be interesting to understand the structure of $\zeta_{G, \epsilon}$ for G p -adic Lie group. Can we apply the same techniques developed in [14]?

Is it possible to have simple closed expression for the computation of the zeta function generalisation in the case of $GL(n, \mathbb{F}_q)$, $PGL(n, \mathbb{F}_q)$, and more generally in the case $GL(n, \mathcal{O})$, $PGL(n, \mathcal{O})$?

Can we generalize the notion of quantum double to the case of compact p -adic Lie groups along the line for example of [18] and define their generalized ζ function representations?

5. ACKNOWLEDGMENTS

I thank I.Badulescu for discussions.

APPENDIX A. TABLE OF CHARACTERS , SCHUR FROBENIUS INDICATOR AND EXPLICIT DECOMPOSITION OF TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF THE GROUPS $GL(2, \mathbb{F}_q)$, $PGL(2, \mathbb{F}_q)$.

A.1. Generalities. We have used the work of J.Adams[1] and we have completed it.

Let $\mathbb{F} = \mathbb{F}_q$ finite field with $q = p^e$ element where p is any prime number and $e \geq 1$. We have $GL(2, \mathbb{F}) = \{A \in Mat(2, \mathbb{F}), \det(A) \in \mathbb{F}^\times\}$. The center is equal to $Z = Z(GL(2, \mathbb{F})) = \mathbb{F}^\times I$, where I is the unit matrix. We denote $PGL(2, \mathbb{F}) = GL(2, \mathbb{F})/Z$. We have $SL(2, \mathbb{F}) = \{A \in Mat(2, \mathbb{F}), \det(A) = 1\}$. The centre $Z(SL(2, \mathbb{F})) = \{\lambda I, \lambda \in \mathbb{F}, \lambda^2 = 1\}$,

and $PSL(2, \mathbb{F}) = SL(2, \mathbb{F})/Z(SL(2, \mathbb{F}))$. Depending on the parity of p , we have $Z(SL(2, \mathbb{F})) = \{I\}$ if $p = 2$ and $Z(SL(2, \mathbb{F})) = \{I, -I\}$ if p is odd.

We have $|GL(2, \mathbb{F})| = q(q-1)^2(q+1)$, $|PGL(2, \mathbb{F})| = q(q-1)(q+1)$, $|SL(2, \mathbb{F})| = q(q-1)(q+1)$ and $|PSL(2, \mathbb{F})| = \frac{q(q-1)(q+1)}{\gcd(2, q-1)}$. As a result when q is odd all these groups are different, when q is even but different of 2 we have $GL(2, \mathbb{F}) \neq PSL(2, \mathbb{F}) = PGL(2, \mathbb{F}) = SL(2, \mathbb{F})$. When $q = 2$ we have $GL(2, \mathbb{F}) = PSL(2, \mathbb{F}) = PGL(2, \mathbb{F}) = SL(2, \mathbb{F}) = S_3$.

We denote $\mathbb{E} = \mathbb{F}_{q^2}$, extension of \mathbb{F} of dimension 2 and denote σ the generator of $Gal(\mathbb{E}, \mathbb{F}) = \mathbb{Z}/2\mathbb{Z}$. σ is the Frobenius element, we have $\sigma(a) = a^q, \forall a \in \mathbb{E}$, and because \mathbb{E} is the unique quadratic extension of \mathbb{F} it will be convenient to denote $\sigma(a) = \bar{a}$. As usual we denote the norm N and the trace Tr being the maps from \mathbb{E} to \mathbb{F} defined by $N(\lambda) = \lambda\bar{\lambda} = \lambda^{q+1}$ and $Tr(\lambda) = \lambda + \bar{\lambda} = \lambda + \lambda^q$.

We have to consider the two cases q even and q odd.

When q is odd the map $\mathbb{F} \rightarrow \mathbb{F}, x \mapsto x^2$ is not surjective. Let Δ a non zero element of \mathbb{F} which is not a square. Let $\delta \in \mathbb{E}$ a square root of Δ . We have $\mathbb{E} = \mathbb{F}(\delta)$, and $\bar{\delta} = \delta^q = -\delta$. As a result if $x \in \mathbb{E}$ we can uniquely write $x = a + b\delta$ with $a, b \in \mathbb{F}$ and $Tr(x) = 2a$ and $N(x) = a^2 - b^2\Delta$.

When q is a power of 2, every element of \mathbb{F} is a square. Let us consider instead the map $\mathbb{F} \rightarrow \mathbb{F}, x \mapsto x^2 + x$. This is a \mathbb{F}_2 linear map, never surjective and its image has $\frac{1}{2}q$ elements. Let Ω be any element which is not in the image of this map. We have $\mathbb{E} = \mathbb{F}(\omega)$ where ω is a root of the irreducible polynomial $X^2 + X + \Omega$.

A.2. Character table of $GL(2, \mathbb{F})$.

A.2.1. Conjugacy classes. We first give a list of the conjugacy classes, which are classified into 4 types, the number of conjugacy classes of this type and the number of element of each conjugacy class. For each of them we choose a particular representative. Two objects will be in relation \sim if and only if their corresponding conjugacy class are equal.

- (Central) $c_1(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}^\times$
Number of classes: $q - 1$, Number of elements in each conjugacy classes: 1
- (Unipotent) $c_2(x) = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, x \in \mathbb{F}^\times$
Number of classes: $q - 1$, Number of elements in each conjugacy classes: $(q - 1)(q + 1)$
- (Diagonal) $c_3(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, (x, y) \in (\mathbb{F}^\times)^2, x \neq y, (x, y) \sim (y, x)$
Number of classes: $\frac{(q-1)(q-2)}{2}$, Number of elements in each conjugacy classes: $q(q + 1)$

- (Elliptic) $c_4(\lambda) = \begin{pmatrix} 0 & -\lambda\bar{\lambda} \\ 1 & \lambda + \bar{\lambda} \end{pmatrix}, \lambda \in \mathbb{E} \setminus \mathbb{F}, \lambda \sim \bar{\lambda}.$

Number of classes: $\frac{q(q-1)}{2}$, Number of elements in each conjugacy classes: $q(q-1)$

A.2.2. Irreducible representations. We now give the list of finite dimensional complex irreducible representations of $GL(2, \mathbb{F})$. The group of one dimensional complex representations of \mathbb{F}^\times is the group of multiplicative characters and is denoted $\widehat{\mathbb{F}^\times}$.

We use the implicit definition of a relation \sim : two objects in a set will be in relation \sim if and only if their corresponding representation are equivalent.

There are 4 types of irreducible representations.

- Let μ be a multiplicative character of \mathbb{F}^\times , we denote χ_μ the one dimensional representation of $G = GL(2, \mathbb{F})$ defined by $\chi_\mu(g) = \mu(\det(g)), \forall g \in G$. These are the one dimensional representations of G , i.e the characters of G . There are $q-1$ inequivalent representations of this type.
- Let T be the diagonal torus and N the unipotent group of upper triangular elements, we denote B the Borel subgroup, we have $B = TN$. Let μ_1, μ_2 be multiplicative characters, we define a one dimensional representations $\mu_1 \otimes \mu_2$ of T , extend it trivially to N to a one dimensional representation still denoted $\mu_1 \otimes \mu_2$ of B . We denote $I(\mu_1, \mu_2) = \text{Ind}_B^G(\mu_1 \otimes \mu_2)$. These representations are of dimension $[G : B] = q + 1$. They are irreducible if and only if $\mu_1 \neq \mu_2$. We have $I(\mu_1, \mu_2) \simeq I(\mu_2, \mu_1)$, therefore $(\mu_1, \mu_2) \sim (\mu_2, \mu_1)$. These representations are the principal representations. There are $\frac{1}{2}(q-1)(q-2)$ inequivalent representations of this type.
- The representation $I(\mu, \mu)$ is not irreducible, it contains χ_μ as a subrepresentation. The representation $St_\mu = I(\mu, \mu)/\chi_\mu$ is irreducible of dimension q . For $\mu = 1$, the representation St_1 is usually called Steinberg representation and denoted St . We have $St_\mu \simeq \chi_\mu \otimes St$. There are $q-1$ inequivalent representations of this type.
- There is a fourth class of representations, more difficult to construct, these are the cuspidal representations. We describe here a simple method of construction of these representations [12]. The map $N : \mathbb{E}^\times \rightarrow \mathbb{F}^\times$ induces an injective map $\widehat{\mathbb{F}^\times} \hookrightarrow \widehat{\mathbb{E}^\times}, \mu \mapsto \mu \circ N$. The elements of $\widehat{\mathbb{E}^\times}$ which are not in the image of this map are called primitive characters. An element ν in $\widehat{\mathbb{E}^\times}$ is primitive if and only if $\bar{\nu} = \nu \circ \sigma = \nu^q \neq \nu$.

\mathbb{E}^\times acts linearly on \mathbb{E} , of dimension 2 over \mathbb{F} , therefore one can identify \mathbb{E}^\times with a subgroup of $GL(2, \mathbb{F})$.

Let Z denote the center of $GL(2, \mathbb{F})$ and let ν be a primitive character of \mathbb{E}^\times , and ψ a non trivial character of the abelian group

N . We denote ν_ψ the character of the subgroup ZN by:

$$\nu_\psi : \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} n \mapsto \nu(a)\psi(n), a \in \mathbb{F}^\times, n \in N.$$

The representation $Ind_{ZN}^G(\nu_\psi)$ is, up to equivalence, independent of ψ , and we have

$$Ind_{ZN}^G(\nu_\psi) = \mathcal{C}_\nu \oplus Ind_{\mathbb{F}^\times}^G(\nu)$$

where \mathcal{C}_ν is an irreducible representation of dimension $q - 1$, the cuspidal representation.

One has $\mathcal{C}_{\nu_1} \simeq \mathcal{C}_{\nu_2}$ if and only if $\nu_2 = \overline{\nu_1}$ or $\nu_2 = \nu_1$. Therefore $\nu \sim \overline{\nu}$. There are $\frac{1}{2}q(q-1)$ inequivalent representations of this type.

The character table is given by the following table which holds for any q power of prime:

	$c_1(x)$	$c_2(x)$	$c_3(x, y)$	$c_4(\lambda)$
χ_μ	$\mu(x)^2$	$\mu(x)^2$	$\mu(x)\mu(y)$	$\mu(\lambda\overline{\lambda})$
$I(\mu_1, \mu_2)$	$(q+1)\mu_1(x)\mu_2(x)$	$\mu_1(x)\mu_2(x)$	$\mu_1(x)\mu_2(y) + \mu_1(y)\mu_2(x)$	0
St_μ	$q\mu(x)^2$	0	$\mu(x)\mu(y)$	$-\mu(\lambda\overline{\lambda})$
\mathcal{C}_ν	$(q-1)\nu(x)$	$-\nu(x)$	0	$-\nu(\lambda) - \nu(\overline{\lambda})$

A.2.3. Frobenius-Schur indicator. We give here the list of Frobenius-Schur indicators of irreducible representations of $GL(2, \mathbb{F})$. Rather than computing these indicators by the formula (14) (which is tedious), we prefer to use the following description. We recall that the Frobenius-Schur indicator of a complex irreducible representation π is the integer $\nu_2(\pi)$ which is defined by (14), it takes the value 0 if the character χ_π is not real, otherwise it takes the value 1 if the representation is isomorphic to a real representation and is equal to -1 if it is not the case. Therefore by inspection of the character table of $GL(2, \mathbb{F})$, it is simple to find the representations which have non zero indicator. We will finally show that there is no irreducible representation having -1 value of the indicator by a counting argument.

- $\nu_2(\chi_\mu)$ is non zero if and only if μ is real. This is possible if and only if $\mu^2 = 1$. The solutions of $\mu^2 = 1$ depend on the parity of q . When q is even there is a unique solution $\mu = 1 = \epsilon_+$, when q is odd one obtains two solutions, the trivial one $\mu = 1 = \epsilon_+$ and the non trivial character ϵ of \mathbb{F}^\times with value in $\{+1, -1\}$. We have $\epsilon(x) = 1$ if and only if x is a square.

Therefore there are two cases:

if q is even then $\nu_2(\chi_\mu) = 0$ if $\mu \neq 1$ and $\nu_2(\chi_1) = 1$

if q is odd then $\nu_2(\chi_\mu) = 0$ if $\mu \notin \{1, \epsilon\}$ and $\nu_2(\chi_1) = \nu_2(\chi_\epsilon) = 1$.

- From the character table, the indicator of $I(\mu_1, \mu_2)$ is non zero if and only if $\mu_1\mu_2$ and $\mu_1 + \mu_2$ are real. There are two cases:
 - if q is even then $\nu_2(I(\mu_1, \mu_2)) = 0$ if and only if $\mu_1\mu_2 \neq 1$.

if q is odd then $\nu_2(I(\mu_1, \mu_2)) = 0$ if $\mu_1\mu_2 \neq 1$ or $\mu_1\mu_2 \neq \epsilon$. If $\mu_1\mu_2 = 1$ then $\nu_2(I(\mu_1, \mu_2)) \neq 0$. If $\mu_1\mu_2 = \epsilon$ then $\nu_2(I(\mu_1, \mu_2)) = 0$ unless $\mu_1 = 1, \mu_2 = \epsilon$ or $\mu_2 = 1, \mu_1 = \epsilon$. We have $\nu_2(I(\epsilon, 1)) = \nu_2(I(1, \epsilon)) \in \{-1, +1\}$.

- $\nu_2(St_\mu)$ is non zero if and only if μ is real. This is possible if and only if $\mu^2 = 1$. Therefore there are two cases:

if q is even then $\nu_2(St_\mu) = 0$ if $\mu \neq 1$ and $\nu_2(St) \in \{+1, -1\}$

if q is odd then $\nu_2(St_\mu) = 0$ if $\mu \notin \{1, \epsilon\}$ and $\nu_2(St) = \nu_2(St_\epsilon) \in \{+1, -1\}$.

- From the character table, the indicator of a cuspidal representation \mathcal{C}_ρ is nul unless $\rho|_{\mathbb{F}^\times}^2 = 1$. It is easy to see that the condition $\rho(\lambda) + \rho(\bar{\lambda})$ real forces $\rho|_{\mathbb{F}^\times} = 1$, and in this case $\rho(\lambda) + \rho(\bar{\lambda})$ is real. Therefore $\nu_2(\mathcal{C}_\rho) = 0$ if $\rho|_{\mathbb{F}^\times} \neq 1$, and belongs to $\{-1, +1\}$ otherwise.

It remains to show that when it is non zero the indicator takes the value 1. Let t be the number of non trivial involutions in G , i.e $t = |\{k \in G, k^2 = e, k \neq e\}|$. We have $1+t = \sum_{\pi \in \hat{G}} \nu_2(\pi) \dim(\pi)$ using relation (15). Therefore to show that there is no representation having indicator of value -1 it is sufficient to show that the following relation holds:

$$(23) \quad 1 + t = \sum_{\pi \in \hat{G}, \nu_2(\pi) \neq 0} \dim(\pi).$$

This is done by a counting argument. By solving explicitly the equation $k^2 = e$ in G we obtain that $t = q^2 - 1$ when q is even and $t = q^2 + q + 1$ when q is odd. It is easy to compute $\sum_{\pi \in \hat{G}, \nu_2(\pi) \neq 0} \dim(\pi)$ from the analysis done above, and we verify that the relation (23) holds.

A.3. Character table of $PGL(2, \mathbb{F})$.

A.3.1. Conjugacy classes.

- Because $c_1(x) = xI$, these conjugacy classes project on the conjugacy class of $I = \tilde{c}_1$ in $PGL(2, \mathbb{F})$. One conjugacy of this type.
- Because $c_2(x)$ is similar to $xc_2(1)$ these conjugacy classes project on the conjugacy class of $\tilde{c}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $PGL(2, \mathbb{F})$. One conjugacy class of this type.
- Because $c_3(x, y) = yc_3(x/y, 1)$, and $x \neq y$ these conjugacy classes project on the conjugacy classes of $PGL(2, \mathbb{F})$ of the type $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \tilde{c}_3(x)$ with $x \neq 1$. We have $x \sim x^{-1}$.

Note at this point that there is a distinction between p odd and p even. When p is 2 the number of conjugacy classes of this type is $\frac{1}{2}(q-2)$, when p is odd the number of conjugacy classes of this type is $\frac{1}{2}(q-1)$.

- $\mu c_4(\lambda)$ is similar to the matrix $c_4(\lambda\mu)$. Therefore to classify the conjugacy classes of $PGL(2, \mathbb{F})$ coming from projection of elements of

type c_4 , it is sufficient to analyze the orbits of \mathbb{F}^\times on $\mathbb{E} \setminus \mathbb{F}$ acting by multiplication. Note also that the $\mathbb{Z}/2\mathbb{Z}$ action $\lambda \mapsto \bar{\lambda}$ gives the same conjugacy class. Therefore the set of conjugacy classes of $PGL(2, \mathbb{F})$ coming from projection of elements of type c_4 , is in bijection with $(\mathbb{Z}/2\mathbb{Z}) \backslash (\mathbb{E} \setminus \mathbb{F}) / \mathbb{F}^\times$.

One has to consider the two cases q even and q odd separately.

When q is even, we have $(\mathbb{E} \setminus \mathbb{F}) / \mathbb{F}^\times \simeq \{\lambda \in \mathbb{E}, \lambda \neq 1, N(\lambda) = 1\}$, this comes from the fact that $\mathbb{F} \rightarrow \mathbb{F}, x \mapsto x^2$ is a bijection in characteristic 2. We can always pick a representative of the conjugacy class in $PGL(2, \mathbb{F})$ of $\begin{pmatrix} 0 & -\lambda\bar{\lambda} \\ 1 & \lambda + \bar{\lambda} \end{pmatrix}$ of the form $\begin{pmatrix} 0 & 1 \\ 1 & \lambda + \lambda^{-1} \end{pmatrix}$ with $\lambda\bar{\lambda} = 1$.

Whatever the value of q , the \mathbb{F} -linear map $Tr : \mathbb{E} \rightarrow \mathbb{F}$ is always a surjection because its kernel $Ker(Tr) = \{\lambda + \lambda^q = 0\}$ has at most q elements.

When q is odd, we can always write $\lambda = x + y\delta$. Therefore in $\mathbb{E} \setminus \mathbb{F}$ there are two types of orbits. The one with $Tr(\lambda) = 0$. In this case $\lambda = y\delta$, and we can select a unique representative in the \mathbb{F}^\times orbit such that $-\lambda\bar{\lambda} = \Delta$, the representative is therefore $\begin{pmatrix} 0 & \Delta \\ 1 & 0 \end{pmatrix}$.

In the case of an orbit such that $Tr(\lambda) \neq 0$, we can choose a unique representative in the \mathbb{F}^\times orbit such that $Tr(\lambda) = 2$. We therefore have for this representative $\lambda = 1 + y\delta, \bar{\lambda} = 1 - y\delta, y \neq 0$. The associated element in $PGL(2, \mathbb{F})$ is $\begin{pmatrix} 0 & -1 + y^2\Delta \\ 1 & 2 \end{pmatrix}$ which is in the same conjugacy class as $\begin{pmatrix} 1 & y\Delta \\ y & 1 \end{pmatrix}$, with $y \neq 0$ and $y \sim y^{-1}$.

A.3.2. Irreducible representations. Let Z the center of G finite group, a representation π of G acting on V is said to admit a central character $\rho : Z \rightarrow \mathbb{C}^\times$ if $\pi(z) = \rho(z)Id_V, \forall z \in Z$. If π admits a central character, we will denote \mathfrak{Z}_π its central character. From Schur lemma, finite dimensional irreducible representations of G admit central character. Moreover if π and π' have central character, $\pi \otimes \pi'$ admits central character $\mathfrak{Z}_{\pi \otimes \pi'} = \mathfrak{Z}_\pi \mathfrak{Z}_{\pi'}$, and $\pi \oplus \pi'$ admits a central character if and only if $\mathfrak{Z}_\pi = \mathfrak{Z}_{\pi'}$.

There is a bijection between representations of $PGL(2, \mathbb{F})$ and representations of $GL(2, \mathbb{F})$ of trivial central character. Therefore the finite dimensional irreducible representations of $PGL(2, \mathbb{F})$ are in bijection with the finite dimensional irreducible representations of $GL(2, \mathbb{F})$ with trivial center character. Therefore the classification of representations of $PGL(2, \mathbb{F})$ is obtained from the character table of $GL(2, \mathbb{F})$ by selecting the irreducible representations of $GL(2, \mathbb{F})$ where the value of $c_1(x)$ is trivial.

One dimensional representations of $PGL(2, \mathbb{F})$ are of the type χ_μ with $\mu^2 = 1$.

The irreducible representations $PGL(2, \mathbb{F})$ belong to the following list:

- χ_μ with $\mu^2 = 1$
- $I(\mu_1, \mu_2)$ with $\mu_1 \mu_2 = 1$.
- St_μ with $\mu^2 = 1$.
- \mathcal{C}_ν with $\nu|_{\mathbb{F}^\times} = id$

When p is 2, the list of inequivalent representations of $PGL(2, \mathbb{F}) \simeq SL(2, \mathbb{F})$ is:

- χ_1 , the trivial representation of dimension 1.
- $I(\mu, 1)$ of dimension $q + 1$ with $\mu \neq 1$, $\mu \sim \mu^{-1}$. There are $\frac{1}{2}q - 1$ inequivalent representations of this type.
- St of dimension q .
- \mathcal{C}_ν of dimension $q - 1$ with $\nu|_{\mathbb{F}^\times} = id$ and $\nu \sim \nu^{-1}$. There are $\frac{1}{2}q$ inequivalent representations of this type.

When p is odd, the list of inequivalent representations of $PGL(2, \mathbb{F})$ is:

- χ_1, χ_ϵ inequivalent representation of dimension 1.
- $I(\mu, 1)$ of dimension $q + 1$ with $\mu \notin \{1, \epsilon\}$, $\mu \sim \mu^{-1}$. There are $\frac{1}{2}(q - 3)$ inequivalent representations of this type.
- $St, St_\epsilon = St \otimes \chi_\epsilon$ inequivalent representations of dimension q .
- \mathcal{C}_ν of dimension $q - 1$ with $\nu|_{\mathbb{F}^\times} = id$ and $\nu \sim \nu^{-1}$. There are $\frac{1}{2}(q - 1)$ inequivalent representations of this type.

Here are the character table of $PGL(2, \mathbb{F})$ in the two cases:

Case where $p = 2$:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, x \sim x^{-1}$	$\begin{pmatrix} 0 & 1 \\ 1 & \lambda + \bar{\lambda} \end{pmatrix}, \bar{\lambda} = \lambda^{-1} \sim \lambda$
χ_1	1	1	1	1
$I(\mu, 1)$	$(q + 1)$	1	$\mu(x) + \mu(x^{-1})$	0
St	q	0	1	-1
\mathcal{C}_ν	$(q - 1)$	-1	0	$-\nu(\lambda) - \nu(\bar{\lambda})$

Case where p is odd:

	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, x \sim x^{-1}$	$\begin{pmatrix} 0 & -\lambda\bar{\lambda} \\ 1 & \lambda + \bar{\lambda} \end{pmatrix}, \bar{\lambda} = \lambda^{-1} \sim \lambda$
χ_1	1	1	1	1
χ_ϵ	1	1	$\epsilon(x)$	$\epsilon(\lambda\bar{\lambda})$
$I(\mu, 1)$	$(q + 1)$	1	$\mu(x) + \mu(x^{-1})$	0
St	q	0	1	-1
St_ϵ	q	0	$\epsilon(x)$	$-\epsilon(\lambda\bar{\lambda})$
\mathcal{C}_ν	$(q - 1)$	-1	0	$-\nu(\lambda) - \nu(\bar{\lambda})$

A.3.3. Frobenius-Schur indicator. There is a bijection between the irreducible representations of $PGL(2, \mathbb{F})$ and the irreducible representations of $GL(2, \mathbb{F})$ having trivial central character. moreover it is easy to show that if $\tilde{\pi}$ is the irreducible representation of $PGL(2, \mathbb{F})$ associated, via this bijection, to the

irreducible representation π of $GL(2, \mathbb{F})$, we have $\nu_2(\pi) = \nu_2(\tilde{\pi})$. From the explicit knowledge of the Frobenius-Schur indicator in the $GL(2, \mathbb{F})$ case, one obtains the value of the Frobenius-Schur indicator in the $PGL(2, \mathbb{F})$ case. We easily obtain that every irreducible representation of $PGL(2, \mathbb{F})$ has Schur-Frobenius indicator equal to 1.

A.4. Decomposition of tensor products of irreducible representations of $GL(2, \mathbb{F})$. We answer here the problem of computing explicitly the multiplicities of irreducible representations in the tensor product of two irreducible representation of $G = GL(2, \mathbb{F})$. This problem is equivalent to the explicit description of the fusion ring of $GL(2, \mathbb{F})$. This is a simple computation, involving only sum over roots of unity, but the explicit result seems not to be known by the experts. Because irreducible representations of $PGL(2, \mathbb{F})$ are in bijection with the subset of irreducible representations of $GL(2, \mathbb{F})$ of trivial central character, the fusion ring of $PGL(2, \mathbb{F})$ is a subring of the the fusion ring of $GL(2, \mathbb{F})$. We therefore study only the structure of the fusion ring of $GL(2, \mathbb{F})$.

If π is an irreducible representation, let $\tilde{\pi}$ be the contragredient representation. Let π, π', π'' irreducible finite dimensional complex representations of G , we denote

$$(24) \quad \langle \pi \pi' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi}(g) \chi_{\pi'}(g),$$

$$(25) \quad \langle \pi \pi' \pi'' \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi}(g) \chi_{\pi'}(g) \chi_{\pi''}(g).$$

The multiplicities are non negative integers which can be computed by the formula

$$(26) \quad N_{\pi, \pi'}^{\pi''} = \frac{1}{|G|} \sum_{g \in G} \chi_{\pi}(g) \chi_{\pi'}(g) \chi_{\pi''}(g^{-1}).$$

We prefer to compute the numbers $N_{\pi, \pi', \pi''} = N_{\pi, \pi'}^{\tilde{\pi}''} = \langle \pi \pi' \pi'' \rangle$ which are symmetric under the exchange of π, π', π'' and for which the explicit formula are simple.

We have

$$(27) \quad \check{\chi}_{\mu} \simeq \chi_{\mu^{-1}}, \check{I}(\mu_1, \mu_2) \simeq I(\mu_1^{-1}, \mu_2^{-1}), \check{S}t_{\mu} \simeq St_{\mu^{-1}}, \check{\mathcal{C}}_{\nu} \simeq \mathcal{C}_{\nu^{-1}}.$$

We will denote δ the delta function on $\widehat{\mathbb{F}^{\times}}$ located on the unit element whereas $\delta^{\mathbb{E}^{\times}}$ denotes the delta function on $\widehat{\mathbb{E}^{\times}}$ located on the unit element. We will denote if $\mu \in \widehat{\mathbb{F}^{\times}}$ and $\nu \in \widehat{\mathbb{E}^{\times}}$, for typographic reasons, $\delta_{\mu} = \delta(\mu)$, $\delta_{\nu}^{\mathbb{E}^{\times}} = \delta^{\mathbb{E}^{\times}}(\nu)$. If ν is a character of \mathbb{E}^{\times} we will also denote $\underline{\nu} = \nu|_{\mathbb{F}^{\times}} \in \widehat{\mathbb{F}^{\times}}$.

We give here the list of non zero elements of $\langle \pi \pi' \rangle$ for π, π' irreducible representations of G ,

Proposition 6.

$$\begin{aligned}
\langle \chi_\mu \chi_\nu \rangle &= \delta_{\mu\nu}, \\
\langle I(\mu_1, \mu_2) I(\nu_1, \nu_2) \rangle &= \delta_{\mu_1 \nu_1} \delta_{\mu_2 \nu_2} + \delta_{\mu_1 \nu_2} \delta_{\mu_2 \nu_1}, \\
\langle St_\mu St_\nu \rangle &= \delta_{\mu\nu}, \\
\langle \mathcal{C}_\nu \mathcal{C}_\rho \rangle &= \delta_{\nu\rho} + \delta_{\nu\bar{\rho}}.
\end{aligned}$$

Proof: it follows from the orthogonality relations on characters and of the stated above equivalence of representations.

Proposition 7.

$$\begin{aligned}
&\langle I(\mu_1, \mu_2) I(\nu_1, \nu_2) I(\rho_1, \rho_2) \rangle = \\
&\delta_{\mu_1 \nu_1 \rho_1 \mu_2 \nu_2 \rho_2} + \delta_{\mu_1 \nu_1 \rho_1} \delta_{\mu_2 \nu_2 \rho_2} + \delta_{\mu_2 \nu_1 \rho_1} \delta_{\mu_1 \nu_2 \rho_2} + \delta_{\mu_1 \nu_2 \rho_1} \delta_{\mu_2 \nu_1 \rho_2} + \delta_{\mu_1 \nu_1 \rho_2} \delta_{\mu_2 \nu_2 \rho_1}, \\
&\langle I(\mu_1, \mu_2) I(\nu_1, \nu_2) St_\rho \rangle = \delta_{\mu_1 \mu_2 \nu_1 \nu_2 \rho^2} + \delta_{\mu_1 \nu_1 \rho} \delta_{\mu_2 \nu_2 \rho} + \delta_{\mu_1 \nu_2 \rho} \delta_{\mu_2 \nu_1 \rho}, \\
&\langle I(\mu_1, \mu_2) I(\nu_1, \nu_2) \mathcal{C}_\rho \rangle = \delta_{\mu_1 \mu_2 \nu_1 \nu_2 \underline{\rho}}, \\
&\langle I(\mu_1, \mu_2) St_\nu St_\rho \rangle = \delta_{\mu_1 \mu_2 \nu^2 \rho^2} + \delta_{\mu_1 \nu \rho} \delta_{\mu_2 \nu \rho}, \\
&\langle St_\mu St_\nu St_\rho \rangle = \delta_{\mu^2 \nu^2 \rho^2}, \\
&\langle I(\mu_1, \mu_2) St_\nu \mathcal{C}_\rho \rangle = \delta_{\mu_1 \mu_2 \nu^2 \underline{\rho}}, \\
&\langle I(\mu_1, \mu_2) \mathcal{C}_\nu \mathcal{C}_\rho \rangle = \delta_{\mu_1 \mu_2 \underline{\nu} \underline{\rho}}, \\
&\langle St_\mu St_\nu \mathcal{C}_\rho \rangle = \delta_{\mu^2 \nu^2 \underline{\rho}}, \\
&\langle St_\mu \mathcal{C}_\nu \mathcal{C}_\rho \rangle = \delta_{\mu^2 \underline{\nu} \underline{\rho}} - \delta_{\mu \bar{\mu} \nu \rho}^{\mathbb{E}^\times} - \delta_{\mu \bar{\mu} \nu \bar{\rho}}^{\mathbb{E}^\times}, \\
&\langle \mathcal{C}_\mu \mathcal{C}_\nu \mathcal{C}_\rho \rangle = \delta_{\underline{\mu} \underline{\nu} \underline{\rho}} - (\delta_{\mu \nu \rho}^{\mathbb{E}^\times} + \delta_{\bar{\mu} \nu \rho}^{\mathbb{E}^\times} + \delta_{\mu \bar{\nu} \rho}^{\mathbb{E}^\times} + \delta_{\mu \nu \bar{\rho}}^{\mathbb{E}^\times}).
\end{aligned}$$

(These last two expressions on the right, although conveniently written with a minus sign, are non negative integer belonging to $\{0, 1\}$).

Proof:

This follows from direct computations. We only give the proof of the first identity, the others follow from the same type of proof.

$$\begin{aligned}
&|G| \langle I(\mu_1, \mu_2) I(\nu_1, \nu_2) I(\rho_1, \rho_2) \rangle = \\
&= \sum_{x \in \mathbb{F}^\times} (q+1)^3 (\mu_1 \mu_2 \nu_1 \nu_2 \rho_1 \rho_2)(x) + \sum_{x \in \mathbb{F}^\times} (\mu_1 \mu_2 \nu_1 \nu_2 \rho_1 \rho_2)(x) (q^2 - 1) + q(q+1) \times \\
&\sum_{\substack{\{x, y\} \subset \mathbb{F}^\times \\ x \neq y}} (\mu_1(x) \mu_2(y) + \mu_1(y) \mu_2(x)) (\nu_1(x) \nu_2(y) + \nu_1(y) \nu_2(x)) (\rho_1(x) \rho_2(y) + \rho_1(y) \rho_2(x)).
\end{aligned}$$

expanding all the product and using the identity

$$(28) \quad \sum_{x \in \mathbb{F}^\times} \mu(x) = (q-1) \delta_\mu, \mu \in \widehat{\mathbb{F}^\times},$$

we obtain the announced result. \square

This is the complete list of $N_{\pi, \pi', \pi''}$ where none of the π, π', π'' are one dimensional representations. Using

$$\begin{aligned}\chi_\mu \otimes \chi_\nu &\simeq \chi_{\mu\nu}, & \chi_\mu \otimes I(\nu, \rho) &\simeq I(\mu\nu, \mu\rho) \\ \chi_\mu \otimes St_\nu &\simeq St_{\mu\nu}, & \chi_\mu \otimes \mathcal{C}_\nu &\simeq \mathcal{C}_{\mu\nu},\end{aligned}$$

we obtain that if π'' is one dimensional, we have $N_{\pi, \pi', \pi''} = \langle \pi(\pi' \otimes \pi'') \rangle$ already computed.

REFERENCES

- [1] J.Adams, "Character tables for $GL(2), SL(2), PGL(2), PSL(2)$, over a finite field", University of Maryland Preprint, (2002).
- [2] L.Aburto-Hageman, J.Pantoja, "Tensor products of irreducible representations of the groups $GL(2, k)$, and $SL(2, k)$, k a finite field", Communications in algebra, 28 (5), 2507-2514, (2000).
- [3] N.Avni, B.Klopsch, U.Onn, C.Voll, "Similarity classes of integral p -adic matrices and representation zeta functions of groups of type A_2 " [arXiv:1410.4533](#)
- [4] I.Badulescu, Ph.Roche, In progress.
- [5] V.Baldoni, A.Boysal, M.Vergne, "Multiple Bernoulli series and volumes of moduli spaces of flat bundle over surfaces," [arXiv:1301.4127](#) [[math.RT](#)]
- [6] L.Brekke, P.O.Freund, "p-adic numbers in physics", Physics Reports, vol 233, Issue 1, 1993.
- [7] B.Bartlett, "Categorical aspects of Topological Quantum field theories", [arXiv:math/0512103](#) [[math.QA](#)]
- [8] R.Dijkgraaf, V.Pasquier, Ph.Roche, "Quasi Hopf algebras, group cohomology and orbifold models", - Nuclear Physics B-Proceedings Supplements, 18 (2) 1991
- [9] R.Dijkgraaf, C.Vafa, E.Verlinde, H.Verlinde, "The operator algebra of orbifold models", Communication in Mathematical Physics, 1989 - Springer.
- [10] R.Dijkgraaf, E.Witten, "Topological gauge theories and group cohomology", Comm.Math.Phys. 129 (1990), 393-429.
- [11] Georg Frobenius and Isaai Schur, "Über die reellen Darstellungen der endlichen Gruppen", Sitzungsberichte der königlich preussischen Akademie der Wissenschaften (1906), 186208.
- [12] G.Henniart, Lectures at the conference $GL(2)$ CIRM (2008).
- [13] N.Ho and L.Jeffrey "The volume of the moduli space of flat connections on a non orientable 2-manifold", [arXiv:math/0307404](#)
- [14] A.Jaikin-Zapirain, "Zeta function representations of compact p -adic analytic group", Journal of the AMS, vol 19, Number 1 2005.
- [15] A.Kable, "Legendre sums, Soto-Andrade sums and Kloosterman sums", Pacific Journal of mathematics, Vol 206, N.1, 2002.
- [16] B.Klopsch, "A newcomer's guide to zeta functions of groups and rings", in Lectures on Profinite Topics in Group Theory, B.Klopsch, N.Nikolov, C.Voll, The London Mathematical Society Student Texts 77 (2011).
- [17] B.Klopsch, "An introduction to compact p -adic Lie groups", in Lectures on Profinite Topics in Group Theory, B.Klopsch, N.Nikolov, C.Voll, The London Mathematical Society Student Texts 77 (2011).
- [18] Koornwinder, Muller, "The quantum double of a (locally) compact group" TH Koornwinder, NM Muller - Journal of Lie Theory, 1997.
- [19] Y.Komoro, K.Matsumoto and H.Tsumura, "Multiple zeta values and zeta functions of root systems", Proc Japan Acad Ser.A Math.Sci 87 n 6 103-107 (2011).
- [20] M.Larsen and A.Lubotzky, "Representation growth of linear groups, J.Eur.Math.Soc (JEMS), 10, 351-390 (2008).

- [21] Yu.I.Manin, "Reflections on arithmetical physics", Conformal invariance and string theory (Poiana, Brasov 1987) 293-303, Academic Press 1989.
- [22] A.A.Migdal, "Recursion equations in gauge field theories", Sov.Phys. JETP, **42**,413 (1975).
- [23] A.D.Mednykh, Determination of the number of non equivalent covering over a compact Riemann surface," Soviet Mathematics Doklady 19 (1978) 318-320.
- [24] E.Meinrenken and C.Woodward, "Modulis spaces of flat connections on 2-manifolds, cobordism and Witten's volume formula, " Advances in geometry, 271295, Progr. Math. 172, Birkhauser, Boston, MA, 1999.**arxiv: dg-ga/9707018** .
- [25] N.Snyder, "Mednykh's formula via lattice topological quantum field theory", **arxiv: math.QA, 0703073**.
- [26] A.Szenes,"Iterated residues and Multiple Bernoulli Polynomials" International Mathematics research Notices 18 937-956 (1998).
- [27] V.Turaev, "Dijkgraaf-Witten invariants of surface and projective representations of groups," **arXiv:0706.0160 [math.GT]**
- [28] C.Voll, " A newcomer's guide to zeta functions of groups and rings", in Lectures on Profinite Topics in Group Theory, B.Klopsch, N.Nikolov, C.Voll, The London Mathematical Society Student Texts 77 (2011).
- [29] C.Voll "Zeta functions of groups and rings - recent developments", Survey based on a talk at the conference Groups St Andrews 2013, **arXiv:1409.7955**
- [30] E.Witten,"On quantum gauge theories in two dimensions". Comm. Math. Phys. 141 153-209 (1991).

PH.ROCHE, UNIVERSITÉ MONTPELLIER 2, CNRS, L2C, IMAG
E-mail address: philippe.roche@univ-montp2.fr